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Estimation of the exponentiated half-logistic distribution under generalized type I hybrid censored samples

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Abstract

In this paper, we consider the shape parameter for the exponentiated half logistic distribution (ExHL) when samples are generalized type I hybrid censored samples. The shape parameter for the ExHL is estimated by the Bayesian method. We consider conjugate prior and corresponding posterior distribution is obtained. We also obtain the maximum likelihood estimator (MLE) of the shape parameter under the generalized type I hybrid censored samples (GenT1HCs). We compare the estimators in the sense of the root mean square error (RMSE). The simulation procedure is repeated 1,000 times for the sample size n = 20, 30, 40 and various generalized type I hybrid censored samples. Finally, a real data set has been analysed for illustrative purpose.

Keywords: Bayesian estimation, exponentiated half-logistic distribution, generalized type I hybrid censoring, Linex loss function, maximum likelihood estimation, squared error loss function.

1. Introduction

Consider a life testing experiment in which n units are put on test. Assume that the life times of n units are independent and identically distributed (i.i.d) as exponentiated half logistic distribution (ExHL) with the cumulative density function (cdf)

$$F(x;\lambda) = \left[\frac{1-\exp(-x/\sigma)}{1+\exp(-x/\sigma)}\right]^{\lambda},\ x>0, \lambda>0, \tag{1.1}$$

and the probability density function (pdf)

$$f(x;\lambda) = \frac{2\lambda\exp(-x/\sigma)}{\sigma[1+\exp(-x/\sigma)]}\left[\frac{1-\exp(-x/\sigma)}{1+\exp(-x/\sigma)}\right]^{\lambda-1},\ x>0, \lambda>0, \tag{1.2}$$

where $\lambda > 0$ is shape parameter.

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Inferences for the half logistic distribution (HL) were discussed by several authors. Kang *et al*. (2008) derived the approximate MLE (ApMLE) and MLE of the scale parameter in a HL based on progressively censored samples. Lee *et al*. (2011) derived the ApMLE of the scale parameter in a HL based on doubly generalized type II hybrid censored samples. Recently, Gwag and Lee (2019) derive the ApMLE and MLE of the parameter in a HL based on unified hybrid censored samples. Cho and Lee (2021) derived the ApMLE and Bayes estimators of the scale parameter in a HL based on generalized adaptive progressive hybrid censored samples. Considering Bayesian reliability estimation, some papers are introduced by Kim (2002), Kim (2003a) and Kim (2003b).

The detail description of the GenT1HCs is described as follows. The integer $k \in \{1, 2, ..., r\}$ is pre-fixed. $T \in (0, \infty)$ is a pre-fixed time point. If the k-th failure occurs after time T, terminate the experiment at $X_{k:n}$; if the th failure occurs before T and the r-th failure occurs after T, terminate at T; if the k-th and r-th failures occur before T, terminate at $X_{r:n}$ (See Chandrasekar *et al*. (2004)).

Let d denote the number of observed failures up to time T. In this scheme, we have one of the following three types of observations;

Case I : $X_{1:n}, X_{2:n}, ..., X_{k:n}$, if $T < X_{k:n} < X_{r:n}$,
Case II : $X_{1:n}, X_{2:n}, ..., X_{d:n}$, if $X_{k:n} < T < X_{r:n}$,
Case III : $X_{1:n}, X_{2:n}, ..., X_{r:n}$, if $X_{k:n} < X_{r:n} < T$.

A schematic representation of the GenT1HCs is presented in Figure 1.1.

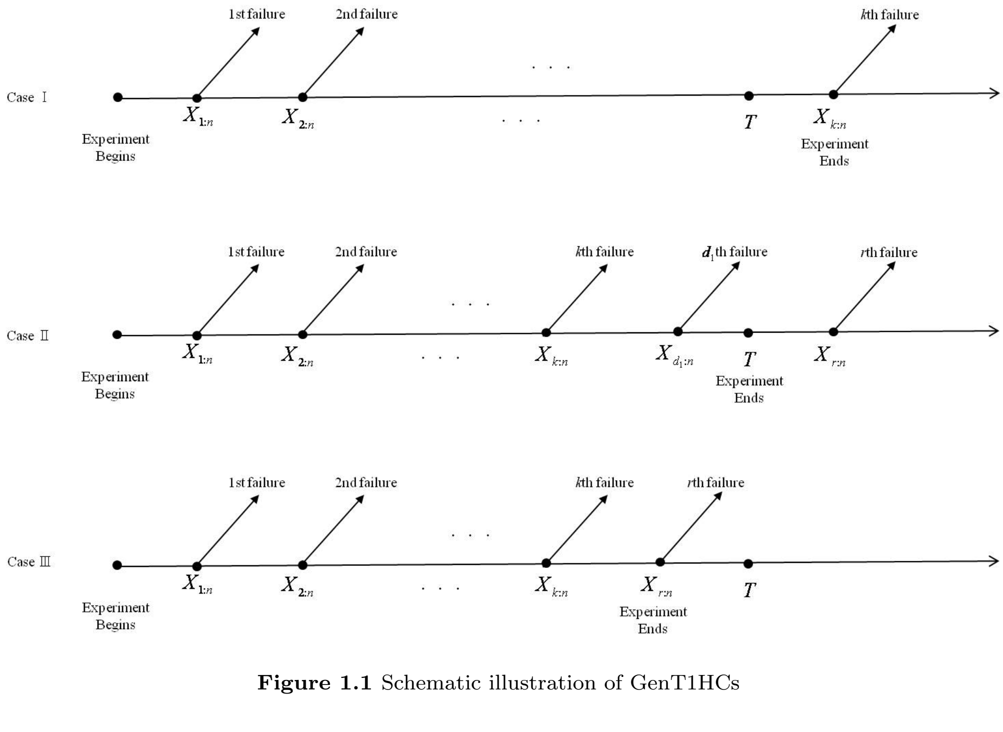

Figure 1.1 Schematic illustration of GenT1HCs

This study is two aims. The 1st is to consider the MLE of the shape parameter when the samples are GenT1HCs. However, MLE cannot be obtained in a closed form. We use the numerical method. The 2nd is to consider the Bayesian estimation for the shape parameter under square error loss function (SqrEL) and linex loss function (LinL).

The rest of this paper is organized as follows. In Section 2, we describe the computation of the MLE of the λ based on the GenT1HCs. In Section 3, Bayes estimators of the λ under the SqrEL and LinL are derived. In Section 4, the description of different estimators that are compared by performing the Monte Carlo simulation is presented, and Section 5 concludes.

2. Maximum likelihood estimation

Assume that the failure times of the units are the ExHL with cdf (1.1) and pdf (1.2). The likelihood functions for three Cases are as follows.

$$L_1(\lambda) \propto \lambda^k \prod_{i=1}^{k} \left[\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right]^{\lambda-1} \left[1-\left\{\frac{1-\exp(-x_{k:n}/\sigma)}{1+\exp(-x_{k:n}/\sigma)}\right\}^{\lambda}\right]^{n-k},$$

$$L_2(\lambda) \propto \lambda^d \prod_{i=1}^{d} \left[\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right]^{\lambda-1} \left[1-\left\{\frac{1-\exp(-T/\sigma)}{1+\exp(-T/\sigma)}\right\}^{\lambda}\right]^{n-d},$$

$$L_3(\lambda) \propto \lambda^r \prod_{i=1}^{r} \left[\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right]^{\lambda-1} \left[1-\left\{\frac{1-\exp(-x_{r:n}/\sigma)}{1+\exp(-x_{r:n}/\sigma)}\right\}^{\lambda}\right]^{n-r}.$$

Cases I, II and III can be combined and be represented as

$$L(\lambda) \propto \lambda^z \prod_{i=1}^{z} \left[\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right]^{\lambda-1} \left[1-\left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z}, \tag{2.1}$$

where $z = k$ and $\xi = x_{k:n}$ for Case I, $z = d$ and $\xi = T$ for Case II, and $z = r$ and $\xi = x_{r:n}$ for Case III. From (2.1), the log-likelihood function can be expressed as

$$\begin{aligned}\log L(\lambda) \propto & z \log\lambda + (\lambda-1)\sum_{i=1}^{z} \log\left[\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right] \\ & + (n-z)\log\left[1-\left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right].\end{aligned}$$

On differentiating the log-likelihood function with the respect to λ and equating to zero, we obtain the estimating equation,

$$\frac{\partial \log L(\lambda)}{\partial\lambda} = \frac{z}{\lambda} + \sum_{i=1}^{z} \log\left[\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right] - (n-z)\frac{\log\left[\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right]}{1-\left[\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right]^{\lambda}} = 0.$$

This equation is in implicit form, so it may be subsequently solved with a numerical method.

3. Bayesian estimation

In Bayesian estimation, we consider two types of loss functions. The first is the SqrEL which is symmetrical. But in life testing problems, the nature of losses are not always symmetric. As an alternative to the SqrEL, the second is the LinL which is asymmetric.

Since based on the GenT1HCs is a random variable, we consider the natural conjugate family of prior distributions for that were used as

$$\pi(\lambda; a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda), \quad a > 0, \ b > 0. \tag{3.1}$$

By combining (2.1) with (3.1), the joint density function of λ and $\boldsymbol{X}$ is given by

$$\begin{aligned}\pi(\lambda, \boldsymbol{X}) \propto & \lambda^{z+a-1} \exp\left[-\lambda\left\{b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right] \\ & \times \left[1 - \left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z}.\end{aligned}$$

Further, the posterior density function of is given by

$$\begin{aligned}\pi(\lambda|\boldsymbol{X}) = & \frac{1}{\Gamma(z+a)K_1(z+a)} \\ & \times \lambda^{z+a-1} \exp\left[-\lambda\left\{b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right] \left[1 - \left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z},\end{aligned}$$

where

$$K_1(t) = \sum_{j=0}^{z} (-1)^j \binom{n-z}{j} \left[b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right) - j \log\left(\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right)\right]^{-t}$$

and

$$\begin{aligned}\int_0^\infty \pi(\lambda, \boldsymbol{X}) d\lambda = & \int_0^\infty \lambda^{z+a-1} \exp\left[-\lambda\left\{b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right] \\ & \times \left[1 - \left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z} d\lambda \\ = & \int_0^\infty \lambda^{z+a-1} \exp\left[-\lambda\left\{b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right] \\ & \times \sum_{j=0}^{z} (-1)^j \binom{n-z}{j} \left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{j\lambda} d\lambda\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{z}(-1)^j \binom{n-z}{j} \int_0^\infty \lambda^{z+a-1} \exp\left[-\lambda\left\{b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right.\right.\\
&\quad \left.\left. -j\log\left(\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right)\right\}\right] d\lambda\\
&= \Gamma(z+a)\sum_{j=0}^{z}(-1)^j \binom{n-z}{j} \left[b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right.\\
&\quad \left. -j\log\left(\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right)\right]^{-(z+a)}\\
&= \Gamma(z+a)K_1(z+a).
\end{aligned}
$$

Under SqrEL, the Bayes estimator of λ is the mean of the posterior density given by

$$
\begin{aligned}
\hat{\lambda}_S &= \int_0^\infty \lambda\pi(\lambda|\boldsymbol{X})d\lambda\\
&= \frac{1}{\Gamma(z+a)K_1(z+a)} \int_0^\infty \lambda^{z+a} \exp\left[-\lambda\left\{b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right]\\
&\quad \times \left[1 - \left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z} d\lambda\\
&= \frac{(z+a)K_1(z+a+1)}{K_1(z+a)}.
\end{aligned}
$$

Under LinL, the Bayes estimator of λ is given by

$$
\hat{\lambda}_L = -\frac{1}{h}\log E(e^{-h\lambda}) = -\frac{1}{h}\log\frac{K_2(z+a)}{K_1(z+a)},
$$

where h is the scale parameter of LinL,

$$
\begin{aligned}
E(e^{-h\lambda}) &= \int_0^\infty e^{-h\lambda}\pi(\lambda|\boldsymbol{X})d\lambda\\
&= \frac{1}{\Gamma(z+a)K_1(z+a)} \int_0^\infty \lambda^{z+a-1} \exp\left[-\lambda\left\{h + b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right]\\
&\quad \times \left[1 - \left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z} d\lambda\\
&= \frac{K_2(z+a)}{K_1(z+a)}
\end{aligned}
$$

and

$$
K_2(t) = \sum_{j=0}^{z}(-1)^j \binom{n-z}{j} \left[h + b - \sum_{i=1}^{z} \log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right) - j\log\left(\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right)\right]^{-t}.
$$

For the situation where no prior information about the shape parameter λ is available, one may use the quasi density as given by

$$\pi(\lambda; c) = \frac{1}{\lambda^c}, \quad c > 0. \tag{3.2}$$

By combining (2.1) with (3.2), the joint density function of λ and $\boldsymbol{X}$ is given by

$$\pi(\lambda, \boldsymbol{X}) \propto \lambda^{z-c} \exp\left[-\lambda\left\{-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right]\left[1-\left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z}.$$

Further, the posterior density function of λ is given by

$$\pi(\lambda|\boldsymbol{X}) = \frac{1}{\Gamma(z-c+1)K_3(z-c+1)}\lambda^{z-c}\exp\left[-\lambda\left\{-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right]$$
$$\times\left[1-\left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z},$$

where

$$K_3(t) = \sum_{j=0}^{z}(-1)^j\binom{n-z}{j}\left[-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right) - j\log\left(\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right)\right]^{-t},$$

and

$$\begin{aligned}
\int_0^\infty \pi(\lambda, \boldsymbol{X})d\lambda =& \int_0^\infty \lambda^{z-c}\exp\left[-\lambda\left\{-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right] \\
&\times\left[1-\left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z} d\lambda \\
=& \int_0^\infty \lambda^{z-c}\exp\left[-\lambda\left\{-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right] \\
&\times\sum_{j=0}^{z}(-1)^j\binom{n-z}{j}\left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{j\lambda} d\lambda \\
=& \sum_{j=0}^{z}(-1)^j\binom{n-z}{j}\int_0^\infty \lambda^{z-c}\exp\left[-\lambda\left\{-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right.\right. \\
&\left.\left.-j\log\left(\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right)\right\}\right] d\lambda \\
=& \Gamma(z-c+1)\sum_{j=0}^{z}(-1)^j\binom{n-z}{j}\left[-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right. \\
&\left.-j\log\left(\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right)\right]^{-(z-c+1)} \\
=& \Gamma(z-c+1)K_3(z-c+1).
\end{aligned}$$

Under SqrEL, the Bayes estimator of λ is the mean of the posterior density given by

$$\begin{aligned}\hat{\lambda}_s =& \int_0^\infty \lambda\pi(\lambda|\boldsymbol{X})d\lambda \\ =& \frac{1}{\Gamma(z-c+1)K_3(z-c+1)}\int_0^\infty \lambda^{z-c+1}\exp\left[-\lambda\left\{-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right] \\ &\times\left[1-\left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z} d\lambda \\ =& \frac{(z-c+1)K_3(z-c+2)}{K_3(z-c+1)}.\end{aligned}$$

Under LinL, the Bayes estimator of λ is given by

$$\hat{\lambda}_L = -\frac{1}{h}\log E(e^{-h\lambda}) = -\frac{1}{h}\log\frac{K_4(z-c+1)}{K_3(z-c+1)},$$

where

$$\begin{aligned}E(e^{-h\lambda}) =& \int_0^\infty e^{-h\lambda}\pi(\lambda|\boldsymbol{X})d\lambda \\ =& \frac{1}{\Gamma(z-c+1)K_3(z-c+1)}\int_0^\infty \lambda^{z-c}\exp\left[-\lambda\left\{h-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)\right\}\right] \\ &\times\left[1-\left\{\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right\}^{\lambda}\right]^{n-z} d\lambda \\ =& \frac{K_4(z-c+1)}{K_3(z-c+1)},\end{aligned}$$

and

$$K_4(t) = \sum_{j=0}^{z}(-1)^j\binom{n-z}{j}\left[h-\sum_{i=1}^{z}\log\left(\frac{1-\exp(-x_{i:n}/\sigma)}{1+\exp(-x_{i:n}/\sigma)}\right)+j\log\left(\frac{1-\exp(-\xi/\sigma)}{1+\exp(-\xi/\sigma)}\right)\right]^{-t}.$$

4. Illustrative example and simulation results

4.1. Illustrative example

Mann and Fertig (1973) give failure times to airplane components subjected to a life test. The samples are generalized type I hybrid censored samples : 13 components were placed on test. For this samples, Balakrishnan and Wong (1991) and Lee *et al.* (2011) indicated that the half logistic distribution provides a satisfactory fit. In this sample, we assume that the underlying distribution of this data is the ExHL based on the three GenT1HCs (i.e., Case I: k = 4, T= 2.0, and r = 10; Case II: k = 4, T= 2.0, and r = 8). For Case I, the Bayes estimators under gamma prior of λ_S = 1.672955, and λ_L = 1.620997 are obtained. Also, the Bayes estimators under quasi prior of λ_S = 1.932297, and λ_L = 1.87217 are obtained. And the MLE of $\lambda = 2.030625$ is obtained. For Case II, the Bayes estimators under gamma prior of $\lambda_S = 1.640206$, and $\lambda_L = 1.590039$ are obtained. Also, the Bayes estimators under quasi prior of $\lambda_S = 1.895549$, and $\lambda_L = 1.837401$ are obtained. And the MLE of $\lambda = 2.106521$ is obtained.

4.2. Simulation results

Table 4.1 The relative RMSEs and biases of estimators with MLE and Bayes estimator under gamma prior

				RMSE (bias)				
n	T	r	k	$\hat{\lambda}$	$\hat{\lambda}_S$	$\hat{\lambda}_L$ $(h=-1.5)$	$\hat{\lambda}_L$ $(h=-.5)$	$\hat{\lambda}_L$ $(h=.5)$
20	1	18	10	.2339(.1176)	.1262(.0235)	.1344(.0348)	.1288(.0272)	.1238(.0199)
			12	.2057(.1055)	.1257(.0234)	.1338(.0346)	.1282(.0271)	.1233(.0198)
			14	.1710(.0814)	.1256(.0233)	.1336(.0345)	.1281(.0269)	.1232(.0197)
		16	10	.2336(.1193)	.1263(.0234)	.1344(.0347)	.1288(.0271)	.1239(.0198)
			12	.2053(.1073)	.1258(.0233)	.1338(.0345)	.1283(.0269)	.1234(.0197)
			14	.1705(.0831)	.1256(.0231)	.1336(.0344)	.1282(.0268)	.1233(.0196)
	1.2	18	10	.1947(.0848)	.1261(.0237)	.1342(.0349)	.1287(.0273)	.1238(.0201)
			12	.1840(.0809)	.1257(.0235)	.1338(.0347)	.1282(.0271)	.1233(.0199)
			14	.1644(.0695)	.1256(.0234)	.1336(.0346)	.1282(.0270)	.1233(.0198)
		16	10	.1945(.0891)	.1262(.0235)	.1342(.0347)	.1287(.0272)	.1238(.0199)
			12	.1838(.0852)	.1257(.0233)	.1338(.0345)	.1282(.0270)	.1234(.0197)
			14	.1641(.0738)	.1257(.0232)	.1337(.0344)	.1282(.0269)	.1233(.0196)
30	1	28	20	.1460(.0786)	.0955(.0129)	.0993(.0199)	.0967(.0152)	.0944(.0106)
			22	.1253(.0599)	.0954(.0129)	.0992(.0198)	.0966(.0152)	.0943(.0106)
			24	.1090(.0401)	.0953(.0131)	.0991(.0200)	.0965(.0154)	.0942(.0108)
		26	20	.1459(.0786)	.0955(.0129)	.0993(.0199)	.0967(.0152)	.0944(.0107)
			22	.1252(.0600)	.0954(.0129)	.0991(.0198)	.0966(.0152)	.0942(.0106)
			24	.1089(.0402)	.0953(.0131)	.0991(.0200)	.0965(.0154)	.0942(.0108)
	1.2	28	20	.1338(.0616)	.0953(.0130)	.0991(.0199)	.0965(.0152)	.0942(.0107)
			22	.1220(.0523)	.0953(.0129)	.0991(.0199)	.0965(.0152)	.0942(.0107)
			24	.1089(.0382)	.0953(.0131)	.0991(.0200)	.0965(.0153)	.0942(.0108)
		26	20	.1336(.0619)	.0953(.0130)	.0991(.0199)	.0965(.0153)	.0942(.0107)
			22	.1219(.0526)	.0953(.0130)	.0991(.0199)	.0965(.0152)	.0942(.0107)
			24	.1087(.0385)	.0953(.0131)	.0991(.0200)	.0965(.0154)	.0942(.0108)
40	1	38	30	.1073(.0523)	.0842(.0109)	.0867(.0160)	.0850(.0125)	.0835(.0092)
			32	.0965(.0374)	.0842(.0110)	.0867(.0161)	.0850(.0127)	.0834(.0093)
			34	.0900(.0258)	.0841(.0110)	.0866(.0161)	.0849(.0126)	.0833(.0093)
		36	30	.1073(.0523)	.0842(.0109)	.0867(.0160)	.0850(.0125)	.0835(.0092)
			32	.0965(.0374)	.0842(.0110)	.0867(.0161)	.0850(.0127)	.0834(.0093)
			34	.0900(.0258)	.0841(.0110)	.0866(.0161)	.0849(.0126)	.0833(.0093)
	1.2	38	30	.1061(.0471)	.0843(.0110)	.0868(.0161)	.0850(.0126)	.0835(.0093)
			32	.0967(.0360)	.0842(.0110)	.0867(.0161)	.0850(.0127)	.0834(.0093)
			34	.0901(.0256)	.0841(.0110)	.0866(.0161)	.0849(.0126)	.0833(.0093)
		36	30	.1061(.0471)	.0842(.0110)	.0868(.0161)	.0849(.0126)	.0835(.0093)
			32	.0967(.0361)	.0842(.0110)	.0867(.0161)	.0850(.0127)	.0834(.0093)
			34	.0901(.0256)	.0841(.0110)	.0866(.0161)	.0849(.0126)	.0833(.0093)

Table 4.2 The relative RMSEs and biases of estimators with MLE and Bayes estimator under quasi prior

				RMSE (bias)				
n	T	r	k	$\hat{\lambda}$	$\hat{\lambda}_S$	$\hat{\lambda}_L$ $(h=-1.5)$	$\hat{\lambda}_L$ $(h=-.5)$	$\hat{\lambda}_L$ $(h=.5)$
20	1	18	10	.2339(.1176)	.1565(.0762)	.1683(.0886)	.1603(.0803)	.1529(.0722)
			12	.2057(.1055)	.1559(.0760)	.1676(.0884)	.1596(.0800)	.1522(.0720)
			14	.1710(.0814)	.1556(.0758)	.1672(.0881)	.1593(.0798)	.1520(.0718)
		16	10	.2336(.1193)	.1565(.0761)	.1683(.0885)	.1603(.0801)	.1529(.0721)
			12	.2053(.1073)	.1559(.0759)	.1676(.0883)	.1596(.0799)	.1522(.0719)
			14	.1705(.0831)	.1556(.0756)	.1672(.0880)	.1593(.0797)	.1520(.0717)
	1.2	18	10	.1947(.0848)	.1563(.0762)	.1680(.0886)	.1601(.0802)	.1527(.0722)
			12	.1840(.0809)	.1558(.0760)	.1675(.0884)	.1595(.0800)	.1522(.0720)
			14	.1644(.0695)	.1556(.0758)	.1672(.0881)	.1529(.0798)	.1520(.0719)
		16	10	.1945(.0891)	.1563(.0761)	.1680(.0884)	.1600(.0801)	.1526(.0721)
			12	.1838(.0852)	.1558(.0759)	.1674(.0882)	.1600(.0801)	.1521(.0719)
			14	.1641(.0738)	.1556(.0757)	.1672(.0880)	.1595(.0799)	.1520(.0717)
30	1	28	20	.1460(.0786)	.1115(.0472)	.1174(.0547)	.1134(.0497)	.1097(.0448)
			22	.1253(.0599)	.1113(.0471)	.1171(.0546)	.1132(.0496)	.1095(.0447)
			24	.1090(.0401)	.1113(.0473)	.1171(.0547)	.1132(.0498)	.1095(.0449)
		26	20	.1459(.0786)	.1115(.0472)	.1174(.0547)	.1134(.0497)	.1097(.0448)
			22	.1252(.0600)	.1113(.0472)	.1171(.0546)	.1132(.0496)	.1095(.0447)
			24	.1089(.0402)	.1113(.0473)	.1171(.0548)	.1132(.0498)	.1095(.0449)
	1.2	28	20	.1338(.0616)	.1113(.0472)	.1172(.0547)	.1132(.0497)	.1095(.0448)
			22	.1220(.0523)	.1113(.0472)	.1171(.0546)	.1320(.0496)	.1095(.0448)
			24	.1089(.0382)	.1113(.0473)	.1171(.0547)	.1132(.0497)	.1095(.0449)
		26	20	.1336(.0619)	.1113(.0473)	.1172(.0547)	.1132(.0497)	.1095(.0449)
			22	.1219(.0526)	.1113(.0472)	.1171(.0546)	.1132(.0497)	.1095(.0448)
			24	.1087(.0385)	.1113(.0473)	.1171(.0547)	.1132(.0497)	.1094(.0449)
40	1	38	30	.1073(.0523)	.0950(.0365)	.0988(.0418)	.0962(.0382)	.0938(.0347)
			32	.0965(.0374)	.0950(.0365)	.0988(.0419)	.0962(.0382)	.0937(.0348)
			34	.0900(.0258)	.0949(.0365)	.0987(.0419)	.0962(.0383)	.0937(.0348)
		36	30	.1073(.0523)	.0950(.0365)	.0988(.0418)	.0961(.0383)	.0938(.0347)
			32	.0965(.0374)	.0950(.0365)	.0988(.0419)	.0962(.0382)	.0937(.0348)
			34	.0900(.0258)	.0949(.0365)	.0987(.0419)	.0962(.0383)	.0937(.0348)
	1.2	38	30	.1061(.0471)	.0950(.0366)	.0989(.0963)	.0383(.0383)	.0938(.0348)
			32	.0967(.0360)	.0950(.0365)	.0988(.0419)	.9620(.0383)	.0937(.0348)
			34	.0901(.0256)	.0949(.0365)	.0987(.0419)	.0961(.0383)	.0937(.0348)
		36	30	.1061(.0471)	.0950(.0366)	.0989(.0419)	.0963(.0383)	.0938(.0348)
			32	.0967(.0361)	.0950(.0365)	.0988(.0419)	.0962(.0383)	.0937(.0348)
			34	.0901(.0256)	.0949(.0365)	.0987(.0419)	.0961(.0383)	.0936(.0348)

To compare the performance of the MLE and Bayes estimators of λ under the SqrEL and LinL, we simulated the RMSEs and biases of all proposed estimators through Monte Carlo simulation method. We consider various n, T, r and K.

The GenT1HCs from the ExHL are generated for sample size $n = 20$, 30, 40 and various GenT1HCs. Using this data, the RMSEs and biases of the Bayes estimators of λ under the SqrEL and the LinL are simulated by the Monte Carlo method based on 1,000 times for sample size $n = 20$, 30, 40 and various GenT1HCs. The simulation results under the gamma prior and the quasi prior are given in Tables 4.2 and 4.3, respectively. As the n, r and k increases, the RMSE and bias of the estimates decreases. As the time T increases, the RMSE and bias of the estimates decreases. The MLE of λ is compared with Bayes estimators under the SqrEL and the LinL in terms of estimated RMSE and bias. The computation of Bayes estimators is better than MLEs. The Bayes estimators under the LinL with $h = .5$ show an overall better performance than their corresponding MLE and Bayes estimators under the SqrEL.

5. Conclusions

In this paper, we consider the shape parameter for the ExHL when samples are GenT1HCs. The shape parameter for the ExHL is estimated by the Bayesian method. We consider conjugate prior and corresponding posterior distribution is obtained. We also obtain the maximum likelihood estimator (MLE) of the shape parameter under the generalized type I hybrid censored samples (GenT1HCs). The MLE of λ is compared with Bayes estimators under the SqrEL and the LinL in terms of estimated RMSE and bias. The computation of Bayes estimators is better than MLEs. The Bayes estimators under the LinL with $h = .5$ show an overall better performance than their corresponding MLE and Bayes estimators under the SqrEL. Although we focused on the parameter estimate of the ExHL based on GenT1HCs, estimation of the parameter from other distributions based on GenT1HCs is of potential interest in future research.

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