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Multivariate GLR control charts for the mean vector and covariance matrix

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Abstract

In statistical process control, we want to detect a change in the process accurately and quickly. The GLR (generalized likelihood ratio) control chart has problems with calculating test statistics. In modern times, however, advances in computing systems have enabled GLR chart test statistics calculation. In this paper, multivariate GLR charts were developed to find changes in the mean vector and covariance matrix. Another problem with the multivariate GLR control chart is that the covariance matrix has various forms. In particular, in order to calculate test statistics on the GLR chart, we need to assume that the determinant of covariance matrix increases, and that these assumptions are often unsatisfactory. To solve this problem, this paper suggested giving the lower limit of the covariance matrix. We showed that the GLR control chart is effective in detecting shifts in mean vector and covariance matrix. Especially, the GLR control chart is effective in detecting a wide range of shifts and the GLR control chart does not require initial parameters.

Keywords: Change point, covariance matrix, GLR control chart, mean vector, multivariate normal distribution, SSATS.

1. Introduction

Statistical process control charts are statistical tools used to detect assignable causes in a process. In the process control, detecting a shift in the mean or variance is important problem. We wish to detect shifts quickly and exactly.

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The traditional control charts - Shewhart, cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts have a strength and a weakness. Shewhart control chart is effective in detecting large shifts, but it is not effective in detecting small shifts, CUSUM chart or EWMA chart are effective in detecting small shifts, but there are not effective in detecting large shifts.

In traditional mutivariate control charts, the Shewhart-type multivariate control chart was developed by Hotelling (1947), Healy (1987), Crosier (1988). Pignatiello and Runger (1990) proposed a multivariate CUSUM control chart. Lowry *et al*. (1992) proposed a multivariate EWMA control chart.

In applications, it is difficult to predict a size of shift. In order to detect various shifts, a combination of two or more control charts is one of options. This option has a good performance over various shifts. Shewhart control chart is effective in detecting large shifts and CUSUM chart is effective in detecting small shifts. Therefore Shewhart control chart is used in a combination with CUSUM control chart. But this option requires many control chart parameters.

Another option is using generalized likelihood ratio (GLR) control chart. This option is effective in detecting various shifts and it does not require many control chart parameters.

In prior study, Reynolds and Lou (2010) developed the GLR control chart of the univariate normal distribution with a shift in mean. Reynolds *et al*. (2013) developed the GLR control chart of the univariate normal distribution with a shift in mean and variance, Choi and Lee (2014) developed the GLR control chart with a sustained shift and a linear drift in the process mean. Han *et al*. (2018) developed a Bernoulli GLR chart based on Bayes estimator.

Wang and Reynolds (2013) developed the GLR control chart of the multivariate normal distribution with a shift in mean vector. Zhow and Cho (2018) developed the GLR control chart of the multivariate normal distribution with a shift in covariance matrix.

The objective of this paper is assumed to be the detection of a shift in mean vector and/or variance-covariance matrix.

2. Multivariate GLR control charts

2.1. Notations and assumptions

Let the p vector $X = (x_1, x_2, \cdots, x_p)'$ be process variable and $X_k = (x_{1k}, x_{2k}, \cdots, x_{pk})'$ represent the observation at the kth sampling time point.

Assume that X has the multivariate normal distribution with mean vector μ and covariance matrix Σ. The covariance matrix Σ is as follows;

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix}.$$

When a process is in control, the distribution of X is $MN(\mu_0, \Sigma_0)$. The in control process mean vector and covariance matrix is as follows;

$$\mu_0 = \begin{pmatrix} \mu_{01} \\ \mu_{02} \\ \vdots \\ \mu_{0p} \end{pmatrix}, \Sigma_0 = \begin{pmatrix} \sigma_{011} & \sigma_{012} & \cdots & \sigma_{01p} \\ \sigma_{021} & \sigma_{022} & \cdots & \sigma_{02p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{0p1} & \sigma_{0p2} & \cdots & \sigma_{0pp} \end{pmatrix}.$$

When a process is out of control, the distribution of X is $MN(\mu_1, \Sigma_1)$. The out of control process mean vector and covariance matrix is as follows;

$$\mu_1 = \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1p} \end{pmatrix}, \Sigma_1 = \begin{pmatrix} \sigma_{111} & \sigma_{112} & \cdots & \sigma_{11p} \\ \sigma_{121} & \sigma_{122} & \cdots & \sigma_{12p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p1} & \sigma_{1p2} & \cdots & \sigma_{1pp} \end{pmatrix}.$$

We assume that the in control values μ_0 and Σ_0 are known, and the out of control values μ_1 and Σ_1 are unknown. And n observations in a sample are taken within a sample. The distribution of sample is changed by a shift in mean vector and/or variance-covariance matrix, a shift has occurred at change point τ_c between time τ and $\tau + 1$. We assume the distribution of τ_c is uniform distribution $U(\tau, \tau + 1)$.

Assume that a shift in occurs at random point τ_c and this means that the mean vector μ_0 changes to the unknown mean vector μ_1, Let the noncentrality parameter δ_1 be

$$\delta_1^2 = (\mu_1 - \mu_0)' \Sigma_0^{-1} (\mu_1 - \mu_0).$$

In this paper, we assume that a shift is a sustained shift.

2.2. Generalized likelihood ratio control charts

Suppose kth observation is in control, the maximum likelihood estimator (MLE) of (μ_0, Σ_0) is easily obtained by maximizing likelihood function of multivariate normal distribution.

On the other hand, if the th observation is in out of control, the likelihood function is

$$
\begin{aligned}
L(\tau, \mu_1, \Sigma_1 | X_1, X_2, \cdots, X_k) &= \prod_{i=1}^{\tau} f(X_i|\mu_0, \Sigma_0) \times \prod_{i=\tau+1}^{k} f(X_i|\mu_1, \Sigma_1) \\
&= (2\pi)^{-pk/2} |\Sigma_0|^{-\tau/2} |\Sigma_1|^{-(k-\tau)/2} \\
&\times exp[-\frac{1}{2}(\sum_{i=1}^{\tau}(X_i - \mu_0)'\Sigma_0^{-1}(X_i - \mu_0))] \\
&\times exp[-\frac{1}{2}(\sum_{i=\tau+1}^{k}(X_i - \mu_1)'\Sigma_1^{-1}(X_i - \mu_1))]. \qquad (2.1)
\end{aligned}
$$

And MLE of (μ_1, Σ_1) is as follows;

$$
\hat{\mu}_{1,\tau,k} = \frac{1}{(k-\tau)} \sum_{i=\tau+1}^{k} X_i. \qquad (2.2)
$$

$$
S_{1,\tau,k} = \frac{1}{(k-\tau)} \sum_{i=\tau+1}^{k} (X_i - \hat{\mu}_{1,\tau,k})(X_i - \hat{\mu}_{1,\tau,k})'. \qquad (2.3)
$$

The change point τ_c is unobserved, we estimate τ instead of $\tau + 1$. Under the assumption that the shift in mean vector and/or variance-covariance matrix occurred before time k. The MLE of τ_c can be estimated by maximizing the likelihood function $L(\tau, \mu_1, \Sigma_1 | X_1, X_2, \cdots, X_k)$.

For testing H_0 : the process is in control versus H_1 : the process is out-of-control, we used the log-likelihood ratio as the test statistic.

$$
\begin{aligned}
R_k &= log \frac{\max_{0 \le \tau < k, \mu_1, \Sigma_1} L(\tau, \mu_1, \Sigma_1 | X_1, X_2, \cdots, X_k)}{L(\infty, \mu_0, \Sigma_0 | X_1, X_2, \cdots, X_k)} \\
&= \max_{0 \le \tau < k} \frac{k-\tau}{2}[tr(S_{0,\tau,k}\Sigma_0^{-1}) - tr(S_{1,\tau,k}\Sigma_1^{-1}) - log\frac{|\hat{\Sigma}_1|}{|\hat{\Sigma}_0|}] \\
&= \frac{(k-\hat{\tau})}{2}[tr(S_{0,\tau,k}\Sigma_0^{-1}) - tr(S_{1,\tau,k}\Sigma_1^{-1}) - log\frac{|\hat{\Sigma}_1|}{|\hat{\Sigma}_0|}]. \qquad (2.4)
\end{aligned}
$$

In the univariate normal distribution, X is normally distributed with the mean μ and the variance σ. (μ_0, σ_0) is in control parameter and (μ_1, σ_1) is out of control parameter. Suppose the number of observation n is small, and σ_1 decreases or μ_1 and σ_1 increase, $\hat{\sigma}_1$ will produce an inflated value of R_k (See Reynolds *et al.* (2013)).

If $|\Sigma_1|$ is closed to 0, then $log\frac{|\Sigma_1|}{|\Sigma_0|}$ is diverge and R_k has inflated value. Therefore we use a linear lower bound that depends on $(k-\tau)$ and n.

$$\hat{\Sigma}_{1,B,\tau,k} = \begin{cases} S_{1,\tau,k} & \text{if } |S_{1,\tau,k} \geq (1-\gamma(k-\gamma)|\Sigma_0| \\ (1-\gamma(k-\gamma)|\Sigma_0| & \text{if } |S_{1,\tau,k} < (1-\gamma(k-\gamma)|\Sigma_0|, \end{cases} \tag{2.5}$$

where B indicates a bound and γ is a turning parameter.

2.3. Performance measurements

When the process is out of control, the ATS (average time to signal) is a useful measure of effectiveness of the control charts. The control chart with the smaller ATS is the better. For simplicity, the ATS is calculated under the assumption that a shift in the process occurred at the starting point of the process. In the application, we assume that a shift occurs after the process starts. So we use the SSATS (steady-state ATS) based on the assumption that the process is in control and the state of process changes to out-of-control at a random time τ. And the SSATS is calculated under the assumption that the control statistic reached its steady state distribution before a random time τ. The SSATS applies a realistic situation in SPC (staistical process control) and the SSATS has better the measurement for measuring the ability of control charts.

In this paper, we choose the control limits so that the in control ATS is approximately 400, each parameter is $n=1$, $p=2$. Also the simulation with 10,000 runs was used.

3. Numerical results

3.1. Introduction

In this paper, X has a bivariate normal process and in control parameter (μ_0, Σ_0) is as follows;

$$\mu_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}.$$

For checking the effect of turning parameter γ and selecting optimal turning parameter, we simulate the SSATS with $\gamma \in (0, 0.001, 0.005)$.

The GLR statistic R_k is inefficient for computing performance because it uses the entire data. So we have a window size m to control the amount of data.

$$R_{k,m} = log\frac{\max_{(0,k-m)\leq\tau<k,\mu_1,\Sigma_1} L(\tau,\mu_1,\Sigma_1|X_1,X_2,\cdots,X_k)}{L(\infty,\mu_0,\Sigma_0|x_1,x_2,\cdots,x_k)}$$

The window size m is 25, 50, 100, 200, 400 and 12,000. If $m = 12,000$ then it means $m = \infty$. In Table 3.1, control limits h_{GLR} are obtained by computing when in control ATS is approximately 400.

Table 3.1 Control limits

a	m					
	25	50	100	200	400	12000
0	7.841	7.9101	7.9609	7.9956	8.0066	8.0066
0.001	7.8908	8.0706	8.1607	8.2786	8.3191	8.324
0.005	8.0403	8.3882	8.7382	8.8027	8.8137	8.8137

We choose parameters to check effectiveness in a small shift and a large shift and the process occurs with a shift in the mean vector and the variance-covariance matrix,

(1) σ_{111} and ρ_{11} do not change but δ_1, σ_{122} increase.

(2) ρ_{11} does not change but $\delta_1, \sigma_{111}, \sigma_{122}$ increase.

(3) σ_{111} does not change but δ_1, σ_{122}, increase and ρ_{11} decreases.

(4) $\delta_1, \sigma_{111}, \sigma_{122}$ increase and ρ_{11} decreases.

3.2. Simulation results

For computing SSATS in different variance-covariance matrix structure and mean vector, parameters $(\delta_1, \sigma_{111}, \sigma_{122}, \rho_{11})$ are changed. And we assume that τ is between 200 and 201.

Table 3.2 gives the numerical results for SSATS when σ_{111} and ρ_{11} are not changed but $\delta_1 = 0.1, 1, 2$ and $\sigma_{122} = 1.21, 4$. Table 3.3 gives the numerical results for SSATS when ρ_{11} is not changed but $\delta_1 = 0.1, 1, 2$ and $\sigma_{111}, \sigma_{122} = 1.21, 4$. Table 3.4 gives the numerical results for SSATS when σ_{111} is not changed but $\delta_1 = 0.1, 1, 2$ and $\sigma_{122} = 1.21, 4$ and $\rho_{11} = 0.4, 0.64$. Table 3.5 gives the numerical results for SSATS when $\delta_1 = 0.1, 1, 2$ and $\sigma_{111}, \sigma_{122} = 1.21, 4$ and $\rho_{11} = 0.4, 0.64$. All tables show SSATS increases from the large turning parameter and the large m has small SSATS. But given the efficiency of the computing process, $m = 200$ is most efficient.

Table 3.2 SSATS when noncentrality parameter and one variance are changed

γ	δ_1	σ_{122}	m = 25	50	100	200	400	12000
0	0.1	1.21	70.08	59.13	53.66	51.78	51.59	51.59
		4	4.64	4.63	4.59	4.56	4.57	4.57
	1	1.21	10.84	10.72	10.64	10.57	10.58	10.58
		4	3.76	3.76	3.74	3.71	3.72	3.72
	2	1.21	6.08	6.1	6.06	6.02	6.03	6.03
		4	3.12	3.13	3.11	3.1	3.1	3.1
0.001	0.1	1.21	71.73	60.27	53.2	51.47	51.62	51.62
		4	4.68	4.71	4.68	4.66	4.68	4.68
	1	1.21	10.97	10.92	10.85	10.82	10.9	10.9
		4	3.8	3.82	3.81	3.81	3.83	3.83
	2	1.21	6.16	6.22	6.2	6.2	6.24	6.24
		4	3.16	3.18	3.18	3.18	3.19	3.19
0.005	0.1	1.21	69.63	57.92	55.82	54.08	53.81	53.81
		4	4.71	4.79	4.91	4.87	4.88	4.88
	1	1.21	10.89	11.06	11.39	11.32	11.33	11.33
		4	3.82	3.91	4.01	3.99	3.99	3.99
	2	1.21	6.2	6.39	6.59	6.57	6.57	6.57
		4	3.19	3.26	3.34	3.33	3.33	3.33

Table 3.3 SSATS when noncentrality parameter and two variances are changed

γ	δ_1	$\sigma_{111}, \sigma_{122}$	m = 25	50	100	200	400	12000
0	0.1	1.21	54.89	48.78	45.47	44.29	44.26	44.26
		4	3.76	3.76	3.75	3.73	3.73	3.73
	1	1.21	10.32	10.21	10.13	10.06	10.08	10.08
		4	3.2	3.2	3.19	3.17	3.17	3.17
	2	1.21	5.92	5.93	5.89	5.85	5.87	5.87
		4	2.74	2.74	2.73	2.72	2.72	2.72
0.001	0.1	1.21	56.49	50.61	46.92	46.52	46.84	46.84
		4	3.8	3.84	3.84	3.83	3.85	3.85
	1	1.21	10.43	10.42	10.37	10.33	10.4	10.4
		4	3.24	3.26	3.26	3.26	3.29	3.29
	2	1.21	5.99	6.05	6.03	6.02	6.06	6.06
		4	2.77	2.8	2.8	2.8	2.82	2.82
0.005	0.1	1.21	57.05	52.96	52.68	51.14	50.95	50.95
		4	3.87	3.97	4.1	4.08	4.09	4.09
	1	1.21	10.58	10.73	11.09	11.01	11.02	11.02
		4	3.28	3.37	3.48	3.46	3.46	3.46
	2	1.21	6.1	6.27	6.49	6.45	6.46	6.46
		4	2.82	2.9	2.99	2.98	2.98	2.98

Table 3.4 SSATS when noncentrality parameter is changed, one variance increases and correlation coefficient decreases

γ	δ_1	σ_{122}	ρ_{11}	m					
				25	50	100	200	400	12000
0	0.1	1.21	0.64	21.43	20.18	19.75	19.57	19.61	19.61
			0.4	9.24	9.14	9.06	9.01	9.02	9.02
		4	0.64	3.8	3.79	3.77	3.75	3.75	3.75
			0.4	3.03	3.03	3.02	3	3.01	3.01
	1	1.21	0.64	8.25	8.23	8.17	8.11	8.12	8.12
			0.4	5.79	5.78	5.75	5.72	5.73	5.73
		4	0.64	3.19	3.19	3.17	3.16	3.16	3.16
			0.4	2.64	2.64	2.63	2.62	2.62	2.62
	2	1.21	0.64	5.19	5.21	5.18	5.15	5.15	5.15
			0.4	4.12	4.13	4.11	4.09	4.09	4.09
		4	0.64	2.71	2.71	2.7	2.69	2.69	2.69
			0.4	2.31	2.31	2.3	2.28	2.29	2.29
0.001	0.1	1.21	0.64	21.77	21.31	22.02	21.84	21.86	21.86
			0.4	9.4	9.54	9.89	9.83	9.85	9.85
		4	0.64	3.79	3.87	3.97	3.94	3.95	3.95
			0.4	3.09	3.14	3.22	3.21	3.22	3.22
	1	1.21	0.64	8.37	8.58	8.92	8.87	8.88	8.88
			0.4	5.88	6.04	6.26	6.23	6.24	6.24
		4	0.64	3.25	3.31	3.39	3.38	3.38	3.38
			0.4	2.69	2.75	2.81	2.8	2.81	2.81
	2	1.21	0.64	5.32	5.48	5.67	5.64	5.65	5.65
			0.4	4.22	4.36	4.5	4.49	4.5	4.5
		4	0.64	2.77	2.83	2.91	2.9	2.9	2.9
			0.4	2.35	2.41	2.47	2.46	2.46	2.46
0.005	0.1	1.21	0.64	21.79	20.65	20.32	20.34	20.48	20.48
			0.4	9.34	9.32	9.28	9.26	9.32	9.32
		4	0.64	3.84	3.86	3.84	3.84	3.85	3.85
			0.4	3.06	3.08	3.07	3.07	3.08	3.08
	1	1.21	0.64	8.35	8.38	8.36	8.36	8.41	8.41
			0.4	5.85	5.9	5.87	5.87	5.91	5.91
		4	0.64	3.23	3.24	3.23	3.22	3.24	3.24
			0.4	2.67	2.68	2.68	2.68	2.69	2.69
	2	1.21	0.64	5.26	5.31	5.3	5.3	5.34	5.34
			0.4	4.18	4.22	4.21	4.23	4.26	4.26
		4	0.64	2.74	2.76	2.76	2.76	2.77	2.77
			0.4	2.33	2.34	2.34	2.35	2.36	2.36

Table 3.5 SSATS when noncentrality parameter is changed, two variances increase and correlation coefficient decreases

γ	δ_1	$\sigma_{111}, \sigma_{122}$	ρ_{11}	m					
				25	50	100	200	400	12000
0	0.1	1.21	0.64	17.58	16.88	16.61	16.49	16.53	16.53
			0.4	8.1	8.05	7.97	7.93	7.95	7.95
		4	0.64	2.13	2.13	2.12	2.11	2.11	2.11
			0.4	2.07	2.07	2.06	2.05	2.06	2.06
	1	1.21	0.64	7.62	7.62	7.58	7.53	7.54	7.54
			0.4	5.3	5.31	5.28	5.26	5.26	5.26
		4	0.64	1.9	1.9	1.89	1.88	1.89	1.89
			0.4	1.9	1.9	1.89	1.88	1.89	1.89
	2	1.21	0.64	4.95	4.96	4.94	4.91	4.92	4.92
			0.4	3.88	3.89	3.87	3.85	3.86	3.86
		4	0.64	1.82	1.82	1.81	1.8	1.81	1.81
			0.4	1.74	1.74	1.74	1.73	1.73	1.73
0.001	0.1	1.21	0.64	18.01	18.04	18.81	18.71	18.73	18.73
			0.4	8.27	8.45	8.72	8.67	8.68	8.68
		4	0.64	2.11	2.15	2.21	2.2	2.21	2.21
			0.4	2.11	2.15	2.21	2.2	2.21	2.21
	1	1.21	0.64	7.79	8	8.3	8.25	8.27	8.27
			0.4	5.42	5.56	5.75	5.73	5.73	5.73
		4	0.64	1.94	1.98	2.03	2.02	2.03	2.03
			0.4	1.94	1.98	2.03	2.02	2.03	2.03
	2	1.21	0.64	5.08	5.23	5.42	5.4	5.41	5.41
			0.4	3.98	4.11	4.24	4.22	4.23	4.23
		4	0.64	1.77	1.81	1.85	1.85	1.85	1.85
			0.4	1.77	1.81	1.85	1.85	1.85	1.85
0.005	0.1	1.21	0.64	17.85	17.36	17.2	17.18	17.32	17.32
			0.4	8.2	8.21	8.17	8.15	8.2	8.2
		4	0.64	2.09	2.11	2.1	2.1	2.11	2.11
			0.4	2.09	2.11	2.1	2.1	2.11	2.11
	1	1.21	0.64	7.75	7.78	7.75	7.76	7.81	7.81
			0.4	5.37	5.43	5.41	5.4	5.44	5.44
		4	0.64	1.92	1.93	1.93	1.93	1.94	1.94
			0.4	1.92	1.93	1.93	1.93	1.94	1.94
	2	1.21	0.64	5.02	5.07	5.06	5.07	5.1	5.1
			0.4	3.93	3.98	3.97	3.98	4	4
		4	0.64	1.76	1.77	1.77	1.77	1.78	1.78
			0.4	1.76	1.77	1.77	1.77	1.78	1.78

4. Conclusions

In this paper, The GLR control chart is effective for detecting shifts in and in terms of SSATS. Especially, as the results of simulations, GLR control chart is effective in detecting a wide range of shifts and the GLR control chart does not require initial parameters.

Because a shift of mean vector and variance-covariance matrix can make an inflated R_k, we consider various variance-covariance matrix structures with a turning parameter γ. As the result of simulations, the larger the turning parameter, the more inefficient the GLR chart.

GLR control chart use a lot of past data. For avoiding inefficiency from large data, we use a window size. As the result of simulations, when m is 200, GLR control chart is effective.

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