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Probability matching priors for the powers of mean and variance in the normal distribution†

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Abstract

This paper considers the noninformative priors for the powers of mean and variance in the normal distribution. The powers of mean and variance include the inverse powers of mean, the coefficient of variation and the inverse of the coefficient of variation, etc. We develop probability matching priors as the noninformative priors. Then we reveal that Jeffreys' prior and the developed matching prior have the different density forms. Based on the general priors including the derived noninformative priors, we investigate the condition that the posterior distributions are proper. We show that the the first order matching prior corresponds very well the frequentist target coverage probabilities than Jeffreys' prior through simulation study, and a real example is illustrated.

Keywords: Matching prior, normal distribution, powers of mean and variance.

1. Introduction

Consider the problem of estimating $1/\mu$ for a random sample of size n from a normal distribution with mean μ and variance σ^2. This problem arises in many areas, such as experimental nuclear physics (Lamanna *et al.*, 1981), the one dimensional special case of the single period control problem (Zellner, 1971; Zaman, 1981a) and estimation of structural parameters of a simultaneous equation (Zellner, 1978; Zaman, 1981b), etc. Also, a related problem is the estimation of the inverse of the coefficient of variation. There are many situations requiring this estimate such as, medical imaging (Dong *et al.* 2005), finance (Baker and Edelman, 1991) and electrical and electronic engineering (Brown *et al.* 2001), etc. The inverse of the coefficient of variation is also used as a descriptive parameter in faculty evaluation studies (Rousseau, 1998) and a detection of robust patterns in the spread of epidemics (Crepey and Barthelemy, 2007).

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Therefore, Withers and Nadarajah (2013) considered the general problem of estimating $\mu^I\sigma^{2J}$ given a sample from normal distribution for I as an integer and J as a real. This paper deals with the development of noninformative priors for the powers of mean and variance, $\mu^I\sigma^{2J}$, in the normal distribution. We consider the priors which one-sided Bayesian credible intervals for $\mu^I\sigma^{2J}$ have the correct frequentist coverage. This matching ensures the correct frequentist coverage approximately, but the matching provides a good coverage for small and moderate sample sizes in many problems.

The idea of the matching prior traces back to the study of Welch and Peers (1963). This matching prior has revived in the work of Stein (1985) and the explicit form of the matching prior under an orthogonal parametrization is obtained by Tibshirani (1989). Mukerjee and Dey (1993) for two-parameters models and Mukerjee and Ghosh (1997) for multi-parameter models developed conditions for higher order matching. Along the same lines, Mukerjee and Reid (1999) proposed a matching prior which matches the alternatives coverage probability. Meanwhile, the reference prior has developed by Bernardo (1979) and has revived in the study of Berger and Bernardo (1989), Ghosh and Mukerjee (1992) and Berger and Bernardo (1992). For multi-parameter problems, Berger and Bernardo (1992) proposed a general algorithm for the development of a reference prior. This reference prior gives a good performance in many applications. Also reference priors and matching priors can give the same results or the different results (Kang *et al.*, 2015; Lee *et al.*, 2016).

The remaining studies are summarized as follows. We consider the first order matching priors for the powers of mean and variance using the orthogonal parameterization in Section 2. Next we show that Jeffreys' prior and the first order matching prior have not the same prior density. In Section 3, we investigate the condition that the posterior distribution based on a general prior which includes Jeffreys' prior and the first order matching prior is proper. In Section 4, the frequentist coverage probabilities under the derived noninformative priors are computed. A real example is illustrated.

2. The probability matching prior

Assume that $\mathbf{X} = (X_1, \cdots, X_n)^T$ is a random sample from a normal population with parameters μ and σ^2. Our interest is the parameter $\theta_1 = \mu^a\sigma^b$ with a as an integer and b as a real. Note that the parameter θ_1 includes the parameters, such as the inverse of expectation, coefficient of variation and the inverse of coefficient of variation, etc. Thus we are interest in the development of the noninformative prior for θ_1.

Suppose that $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_t)^T$ is the parameter vector. Then θ_1 is the parameter of interest, while $\theta_2, \cdots, \theta_t$ are nuisance parameters. Also $\theta_1^{1-\alpha}(\pi; \mathbf{X})$ is the $(1-\alpha)$th posterior quantile of θ_1 under the prior π. We consider the rth-order probability matching prior π which satisfies the probability matching constraint

$$P[\theta_1 \le \theta_1^{1-\alpha}(\pi; \mathbf{X})|\boldsymbol{\theta}] = 1 - \alpha + o(n^{-\frac{r}{2}}), r > 0, \tag{2.1}$$

with $r = 1, 2$. Thus the matching priors π with $r = 1$ and $r = 2$ correspond to a first order matching prior and a second order matching prior, respectively.

We consider the development of the matching priors for θ_1. Let

$$\theta_1 = \mu^a\sigma^b \text{ and } \theta_2 = b\mu^2 - 2a\sigma^2, \tag{2.2}$$

where a is an integer and b is a real. Then we compute the Jacobian matrix of the transformation (2.2) and the Jacobian matrix is

$$\frac{\partial(\theta_1,\theta_2)}{\partial(\mu,\sigma)} = \begin{pmatrix} a\mu^{a-1}\sigma^b & b\mu^a\sigma^{b-1} \\ 2b\mu & -4a\sigma \end{pmatrix}. \tag{2.3}$$

Thus from (2.3), the inverse of the expected Fisher information matrix is obtained and is as follow.

$$\begin{aligned} I^{-1}(\theta_1,\theta_2) &= \left(\frac{\partial(\theta_1,\theta_2)}{\partial(\mu,\sigma^2)}\right) I^{-1}(\mu,\sigma^2) \left(\frac{\partial(\theta_1,\theta_2)}{\partial(\mu,\sigma^2)}\right)^t \\ &= \begin{pmatrix} \frac{2a^2\sigma^2+b^2\mu^2}{2\mu^{2-2a}\sigma^{-2b}} & 0 \\ 0 & 4\sigma^2(2a^2\sigma^2+b^2\mu^2) \end{pmatrix}. \end{aligned} \tag{2.4}$$

Therefore from (2.4), the Fisher information matrix is computed and is given by

$$I(\theta_1,\theta_2) = \begin{pmatrix} \frac{2\mu^{2-2a}\sigma^{-2b}}{2a^2\sigma^2+b^2\mu^2} & 0 \\ 0 & \frac{1}{4\sigma^2(2a^2\sigma^2+b^2\mu^2)} \end{pmatrix}. \tag{2.5}$$

Thus from (2.5), θ_1 and θ_2 are orthogonal (Cox and Reid, 1987). Therefore under the orthogonal parametrization, the first order probability matching priors by Tibshirani(1989) is of the form

$$\pi_m(\theta_1,\theta_2) \propto \left(\frac{\mu^{2-2a}\sigma^{-2b}}{2a^2\sigma^2+b^2\mu^2}\right)^{\frac{1}{2}} g(\theta_2), \tag{2.6}$$

where $g(\theta_2) > 0$ is any smooth function of θ_2.

Next Jeffreys' prior for the (θ_1,θ_2) can be obtained by the determination of the Fisher information (2.5) and is given by

$$\pi_J(\theta_1,\theta_2) \propto \left[\frac{\mu^{2-2a}\sigma^{-2b-2}}{(2a^2\sigma^2+b^2\mu^2)^2}\right]^{\frac{1}{2}}. \tag{2.7}$$

Remark 2.1 Notice that we can have many kinds of the matching priors (2.6) in accordance with the choice of the function g. However according to the choice of the function g, the coverage probabilities do not actually seem to be improved. Thus we choose the constant function, and by this choice the matching prior has the simple form. Then the matching prior is of the form

$$\pi_m(\theta_1,\theta_2) \propto \left(\frac{\mu^{2-2a}\sigma^{-2b}}{2a^2\sigma^2+b^2\mu^2}\right)^{\frac{1}{2}}. \tag{2.8}$$

Remark 2.2 In the original parametrization (μ,σ), the matching prior (2.8) is characterized by

$$\pi_m(\mu,\sigma) \propto \sigma^{-1}(2a^2\sigma^2+b^2\mu^2)^{\frac{1}{2}}. \tag{2.9}$$

Also with reference to original parametrization, Jeffreys' prior (2.7) is of the form

$$\pi_J(\mu,\sigma) \propto \sigma^{-2}. \tag{2.10}$$

Remark 2.3 Note that Jeffreys' prior (2.10) is not a first order matching prior (2.9).

3. Propriety of the posterior distribution

We can represent Jeffreys' prior (2.10) and the first order matching prior to general form (2.9) as follows.

$$\pi(\mu,\sigma) \propto \sigma^{-c}(2a^2\sigma^2 + b^2\mu^2)^d, \tag{3.1}$$

where $c > 0$ and $d \geq 0$. For this general prior, we check the condition of the properiety of the posterior distributions. Thus we can obtain the following result.

Theorem 1. The condition of the propriety for the posterior distribution using the general prior (3.1) is $n + c - 2d - 2 > 0$.

Proof. For a given prior (3.1), the joint posterior distribution of θ_1 and θ_2 is of the form

$$\pi(\mu,\sigma|\mathbf{x}) \propto \sigma^{-n-c}(2a^2\sigma^2 + b^2\mu^2)^d \exp\left\{-\frac{1}{2\sigma^2}\left[s^2 + n(\bar{x}-\mu)^2)\right]\right\}, \tag{3.2}$$

where $s^2 = \sum_{i=1}^n (x_i - \bar{x})^2$ and $\bar{x} = \sum_{i=1}^n x_i/n$. Then we can obtain the following equation

$$\begin{aligned}
\pi(\mu,\sigma|\mathbf{x}) &\leq \sigma^{-n-c}(k_1\sigma^{2d} + k_2\mu^{2d})\exp\left\{-\frac{1}{2\sigma^2}\left[s^2 + n(\bar{x}-\mu)^2)\right]\right\} \\
&= k_1\sigma^{-n-c+2d}\exp\left\{-\frac{1}{2\sigma^2}\left[s^2 + n(\bar{x}-\mu)^2\right]\right\} \\
&\quad + k_2\mu^{2d}\sigma^{-n-c}\exp\left\{-\frac{1}{2\sigma^2}\left[s^2 + n(\bar{x}-\mu)^2\right]\right\},
\end{aligned} \tag{3.3}$$

where k_1 and k_2 are constants. Thus

$$\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty \pi(\mu,\sigma|\mathbf{x})d\mu d\sigma &\leq \int_0^\infty \int_{-\infty}^\infty k_1\sigma^{-n-c+2d}\exp\left\{-\frac{s^2 + n(\bar{x}-\mu)^2}{2\sigma^2}\right\}d\mu d\sigma \\
&\quad + \int_0^\infty \int_{-\infty}^\infty k_2\mu^{2d}\sigma^{-n-c}\exp\left\{-\frac{s^2 + n(\bar{x}-\mu)^2}{2\sigma^2}\right\}d\mu d\sigma \\
&= \int_0^\infty \int_{-\infty}^\infty k_1\sigma^{-n-c+2d}\exp\left\{-\frac{s^2 + n(\bar{x}-\mu)^2}{2\sigma^2}\right\}d\mu d\sigma \\
&\quad + \int_0^\infty \int_{-\infty}^\infty k_2(\sigma\omega + \bar{x})^{2d}\sigma^{-n-c+1}\exp\left\{-\frac{s^2}{2\sigma^2} - \frac{n\omega}{2}\right\}d\omega d\sigma.
\end{aligned} \tag{3.4}$$

Therefore the right integrals of (3.4) are proper if $n + c - 2d - 2 > 0$. □

In Section 4, we want to compare Jeffreys' prior π_J and the matching prior π_m about the frequentist coverage probabilities. However we can not obtain the marginal posterior distribution of θ_1, and so Markov chain Monte Carlo (MCMC) method is used to estimate the coverage probability and the marginal moments.

4. Numerical studies

4.1. Simulation study

We investigate the performance of Jeffreys' prior π_J and the matching prior π_m about the frequentist coverage probabilities. By investigating the credible interval for the marginal posterior of $\theta_1 = \mu^a\sigma^b$, we can estimate the frequentist coverage probability for several simulation configurations of (a, b), (μ, σ) and n. That is, we want to know that the frequentist coverage of a $(1-\alpha)$th posterior quantile matches $1-\alpha$. Since the posterior distribution under the developed noninformative priors have not the closed form, thus we compute frequentist coverage probability by the MCMC numerical method. The detailed implementations for the MCMC method are given as follows.

For Jeffreys' prior π_J, we can easily derive the conditional posteriors of μ and σ. However for the matching prior π_m, the conditional posteriors of μ and σ have not the closed forms, and so we give the detailed conditional posteriors as follows.

The joint posterior of μ and σ given $\mathbf{x}$ under the general prior is

$$\pi(\mu,\sigma|\mathbf{x}) \propto \sigma^{-n-c}(2a^2\sigma^2+b^2\mu^2)^d \exp\left\{-\frac{1}{2\sigma^2}\left[s^2+n(\bar{x}-\mu)^2)\right]\right\}.$$

Thus for the matching prior, the full conditionals are given by

$$(\mu|\sigma,\mathbf{x}) \quad \propto \quad (2a^2\sigma^2+b^2\mu^2)^{\frac{1}{2}} \exp\left\{-\frac{n(\bar{x}-\mu)^2)}{2\sigma^2}\right\}, \tag{4.1}$$

$$(\sigma|\mu,\mathbf{x}) \quad \propto \quad \sigma^{-n-1}(2a^2\sigma^2+b^2\mu^2)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left[s^2+n(\bar{x}-\mu)^2)\right]\right\}. \tag{4.2}$$

Under the matching prior, the above conditionals of μ and σ have not standard distribution. Thus we use the Metropolis-Hasting algorithm along the lines of Chib and Greenberg (1995) for sampling from these conditionals.

In each simulation configuration, we generate $20,000$ samples (discarding the first 10,000) from conditionals. Next from each generated samples, we calculate the $\theta_1 = \mu^a\sigma^b$. Lastly, from the computed values of θ_1, we obtain the 5% and 95% posterior quantiles of θ_1. This simulation process is repeated $10,000$ times, and then we calculate the proportion that the true θ_1 belong to this interval. This proportion is the estimate for the frequentist coverage probability of the Bayesian credible interval.

The estimated values for the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles are given in Table 4.1, 4.2, 4.3 and 4.4. From the results of Table 4.1, 4.2, 4.3 and 4.4, we know that the matching prior π_m performs better than than Jeffreys' prior π_J in matching the target coverage probabilities. This is intuitively clear since π_m is a first order matching prior.

Note that it appears from our results that when θ_1 with $(a, b) = (-1, 1)$ becomes larger, the quantile points $\theta_1^{1-\alpha}(\pi_J; \mathbf{x})$ and $\theta_1^{1-\alpha}(\pi_m; \mathbf{x})$ on an average in the frequentist sense become small yielding thereby poor frequentist coverage probabilities. But $\theta_1^{1-\alpha}(\pi_m; \mathbf{x})$ seems to adjust itself when the true θ_1 becomes smaller, and yields good frequentist coverage probabilities (see Gleser and Hwang (1987), and Yin and Ghosh (2001)). Overall, we suggest to use the matching prior for inference of the powers of mean and variance.

4.2. A real example

This example is taken from Fung and Tsang (1998). The Hong Kong Medical Technology Association has conducted the Quality Assurance Programme for medical laboratories in Hong Kong since 1989 and the aim of the programme is to promote the quality and standards

Table 4.1 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

(a, b)	μ	σ	n	π_J	π_m
(-1,1)	1.0	0.2	5	0.025 (0.895)	0.049 (0.953)
			10	0.032 (0.918)	0.052 (0.951)
			15	0.034 (0.922)	0.050 (0.948)
			20	0.038 (0.929)	0.052 (0.950)
		0.5	5	0.000 (0.894)	0.000 (0.956)
			10	0.037 (0.913)	0.054 (0.949)
			15	0.037 (0.926)	0.051 (0.950)
			20	0.040 (0.933)	0.052 (0.952)
		0.7	5	0.000 (0.894)	0.000 (0.955)
			10	0.021 (0.924)	0.008 (0.954)
			15	0.041 (0.931)	0.051 (0.953)
			20	0.041 (0.934)	0.050 (0.953)
		1.0	5	0.000 (0.905)	0.000 (0.963)
			10	0.000 (0.922)	0.000 (0.952)
			15	0.000 (0.931)	0.000 (0.952)
			20	0.032 (0.938)	0.033 (0.954)
(1,1)	1.0	0.2	5	0.022 (0.884)	0.045 (0.951)
			10	0.032 (0.910)	0.054 (0.951)
			15	0.033 (0.925)	0.051 (0.954)
			20	0.035 (0.922)	0.049 (0.945)
		0.5	5	0.011 (0.872)	0.019 (0.958)
			10	0.022 (0.905)	0.041 (0.951)
			15	0.027 (0.916)	0.042 (0.954)
			20	0.029 (0.917)	0.045 (0.947)
		0.7	5	0.010 (0.876)	0.013 (0.964)
			10	0.020 (0.905)	0.035 (0.956)
			15	0.027 (0.913)	0.042 (0.951)
			20	0.029 (0.918)	0.042 (0.953)
		1.0	5	0.013 (0.881)	0.015 (0.971)
			10	0.020 (0.914)	0.029 (0.964)
			15	0.030 (0.917)	0.042 (0.958)
			20	0.032 (0.920)	0.044 (0.953)
(-2,2)	1.0	0.2	5	0.111 (0.972)	0.051 (0.943)
			10	0.080 (0.965)	0.048 (0.949)
			15	0.076 (0.964)	0.051 (0.951)
			20	0.075 (0.966)	0.054 (0.952)
		0.5	5	0.102 (0.965)	0.043 (0.940)
			10	0.079 (0.965)	0.049 (0.949)
			15	0.070 (0.960)	0.049 (0.947)
			20	0.068 (0.957)	0.050 (0.948)
		0.7	5	0.102 (0.963)	0.040 (0.944)
			10	0.078 (0.958)	0.045 (0.946)
			15	0.073 (0.960)	0.050 (0.950)
			20	0.066 (0.963)	0.048 (0.954)
		1.0	5	0.098 (0.965)	0.038 (0.963)
			10	0.072 (0.957)	0.044 (0.951)
			15	0.066 (0.956)	0.047 (0.950)
			20	0.066 (0.958)	0.050 (0.952)

Table 4.2 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

(a,b)	μ	σ	n	π_J	π_m
(-1,1)	10.0	1.0	5	0.025 (0.889)	0.049 (0.948)
			10	0.030 (0.912)	0.051 (0.949)
			15	0.034 (0.927)	0.047 (0.948)
			20	0.034 (0.931)	0.050 (0.952)
		3.0	5	0.028 (0.890)	0.049 (0.956)
			10	0.034 (0.921)	0.052 (0.953)
			15	0.036 (0.925)	0.049 (0.952)
			20	0.037 (0.933)	0.051 (0.951)
		5.0	5	0.000 (0.895)	0.000 (0.957)
			10	0.036 (0.916)	0.053 (0.951)
			15	0.039 (0.926)	0.053 (0.953)
			20	0.040 (0.932)	0.049 (0.951)
		10.0	5	0.000 (0.901)	0.000 (0.959)
			10	0.000 (0.925)	0.000 (0.956)
			15	0.000 (0.932)	0.000 (0.951)
			20	0.035 (0.941)	0.036 (0.955)
(1,1)	10.0	1.0	5	0.024 (0.883)	0.047 (0.950)
			10	0.032 (0.910)	0.050 (0.948)
			15	0.036 (0.923)	0.051 (0.950)
			20	0.039 (0.932)	0.054 (0.951)
		3.0	5	0.017 (0.878)	0.038 (0.954)
			10	0.025 (0.911)	0.048 (0.953)
			15	0.032 (0.918)	0.051 (0.948)
			20	0.035 (0.924)	0.053 (0.950)
		5.0	5	0.011 (0.878)	0.016 (0.962)
			10	0.020 (0.904)	0.036 (0.955)
			15	0.027 (0.917)	0.044 (0.955)
			20	0.029 (0.924)	0.045 (0.953)
		10.0	5	0.014 (0.877)	0.014 (0.969)
			10	0.023 (0.910)	0.034 (0.961)
			15	0.026 (0.922)	0.038 (0.956)
			20	0.031 (0.927)	0.043 (0.957)
(-2,2)	10.0	1.0	5	0.111 (0.974)	0.046 (0.949)
			10	0.086 (0.969)	0.051 (0.949)
			15	0.074 (0.965)	0.049 (0.948)
			20	0.069 (0.961)	0.050 (0.947)
		3.0	5	0.110 (0.971)	0.045 (0.941)
			10	0.081 (0.966)	0.049 (0.947)
			15	0.076 (0.966)	0.051 (0.950)
			20	0.068 (0.962)	0.048 (0.948)
		5.0	5	0.107 (0.969)	0.041 (0.944)
			10	0.085 (0.963)	0.049 (0.948)
			15	0.073 (0.963)	0.048 (0.949)
			20	0.069 (0.961)	0.050 (0.951)
		10.0	5	0.100 (0.962)	0.039 (0.961)
			10	0.072 (0.957)	0.046 (0.950)
			15	0.072 (0.960)	0.051 (0.954)
			20	0.062 (0.958)	0.047 (0.952)

Table 4.3 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

(a,b)	μ	σ	n	π_J	π_m
(-1,1)	-1.0	0.2	5	0.108 (0.972)	0.049 (0.946)
			10	0.078 (0.966)	0.047 (0.948)
			15	0.072 (0.963)	0.049 (0.949)
			20	0.073 (0.965)	0.053 (0.951)
		0.5	5	0.102 (1.000)	0.043 (1.000)
			10	0.080 (0.967)	0.047 (0.952)
			15	0.072 (0.961)	0.048 (0.949)
			20	0.068 (0.958)	0.047 (0.948)
		0.7	5	0.103 (1.000)	0.041 (1.000)
			10	0.079 (0.979)	0.048 (0.992)
			15	0.068 (0.960)	0.046 (0.950)
			20	0.065 (0.960)	0.049 (0.951)
		1.0	5	0.099 (1.000)	0.041 (1.000)
			10	0.074 (1.000)	0.044 (1.000)
			15	0.070 (1.000)	0.049 (1.000)
			20	0.059 (0.965)	0.047 (0.964)
(1,1)	-1.0	0.2	5	0.117 (0.977)	0.051 (0.954)
			10	0.092 (0.972)	0.050 (0.949)
			15	0.074 (0.969)	0.050 (0.954)
			20	0.071 (0.968)	0.047 (0.950)
		0.5	5	0.125 (0.990)	0.039 (0.982)
			10	0.087 (0.977)	0.042 (0.957)
			15	0.084 (0.974)	0.044 (0.955)
			20	0.079 (0.971)	0.047 (0.954)
		0.7	5	0.126 (0.988)	0.036 (0.985)
			10	0.097 (0.978)	0.044 (0.964)
			15	0.084 (0.974)	0.045 (0.959)
			20	0.079 (0.969)	0.046 (0.954)
		1.0	5	0.122 (0.985)	0.030 (0.986)
			10	0.094 (0.979)	0.040 (0.969)
			15	0.082 (0.973)	0.044 (0.962)
			20	0.070 (0.969)	0.041 (0.957)
(-2,2)	-1.0	0.2	5	0.116 (0.976)	0.049 (0.950)
			10	0.086 (0.967)	0.051 (0.949)
			15	0.073 (0.962)	0.049 (0.948)
			20	0.067 (0.962)	0.047 (0.947)
		0.5	5	0.109 (0.964)	0.045 (0.935)
			10	0.078 (0.964)	0.047 (0.947)
			15	0.076 (0.959)	0.052 (0.949)
			20	0.066 (0.959)	0.051 (0.947)
		0.7	5	0.094 (0.965)	0.037 (0.948)
			10	0.080 (0.963)	0.047 (0.951)
			15	0.068 (0.960)	0.047 (0.950)
			20	0.066 (0.961)	0.049 (0.953)
		1.0	5	0.096 (0.964)	0.037 (0.963)
			10	0.072 (0.957)	0.044 (0.950)
			15	0.066 (0.957)	0.047 (0.951)
			20	0.060 (0.956)	0.045 (0.950)

Table 4.4 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

(a,b)	μ	σ	n	π_J	π_m
(-1,1)	-10.0	1.0	5	0.109 (0.977)	0.050 (0.951)
			10	0.089 (0.970)	0.051 (0.951)
			15	0.075 (0.967)	0.052 (0.951)
			20	0.069 (0.965)	0.051 (0.951)
		3.0	5	0.109 (0.971)	0.045 (0.953)
			10	0.084 (0.965)	0.050 (0.947)
			15	0.075 (0.969)	0.051 (0.954)
			20	0.066 (0.963)	0.047 (0.949)
		5.0	5	0.107 (1.000)	0.042 (1.000)
			10	0.082 (0.962)	0.051 (0.951)
			15	0.072 (0.960)	0.048 (0.947)
			20	0.073 (0.961)	0.053 (0.949)
		10.0	5	0.098 (1.000)	0.040 (1.000)
			10	0.076 (1.000)	0.049 (1.000)
			15	0.066 (1.000)	0.045 (1.000)
			20	0.065 (0.967)	0.048 (0.964)
(1,1)	-10.0	1.0	5	0.115 (0.976)	0.052 (0.954)
			10	0.091 (0.970)	0.053 (0.950)
			15	0.078 (0.964)	0.050 (0.948)
			20	0.069 (0.961)	0.047 (0.948)
		3.0	5	0.120 (0.983)	0.045 (0.967)
			10	0.092 (0.974)	0.050 (0.953)
			15	0.081 (0.972)	0.051 (0.954)
			20	0.077 (0.966)	0.051 (0.949)
		5.0	5	0.125 (0.991)	0.039 (0.984)
			10	0.095 (0.976)	0.046 (0.960)
			15	0.086 (0.971)	0.047 (0.954)
			20	0.076 (0.974)	0.047 (0.956)
		10.0	5	0.120 (0.989)	0.030 (0.989)
			10	0.091 (0.977)	0.037 (0.965)
			15	0.085 (0.975)	0.044 (0.963)
			20	0.078 (0.971)	0.046 (0.958)
(-2,2)	-10.0	1.0	5	0.112 (0.976)	0.048 (0.951)
			10	0.086 (0.967)	0.050 (0.948)
			15	0.071 (0.964)	0.047 (0.948)
			20	0.071 (0.962)	0.050 (0.947)
		3.0	5	0.106 (0.975)	0.044 (0.949)
			10	0.079 (0.968)	0.043 (0.950)
			15	0.075 (0.968)	0.051 (0.954)
			20	0.071 (0.963)	0.048 (0.952)
		5.0	5	0.102 (0.970)	0.040 (0.945)
			10	0.084 (0.964)	0.050 (0.949)
			15	0.072 (0.957)	0.047 (0.942)
			20	0.068 (0.961)	0.051 (0.950)
		10.0	5	0.099 (0.961)	0.039 (0.959)
			10	0.077 (0.957)	0.051 (0.951)
			15	0.067 (0.956)	0.048 (0.950)
			20	0.063 (0.956)	0.049 (0.950)

Table 4.5 Estimate and confidence interval for θ_1

θ_1	W-N method	π_J	π_m
σ/μ	0.0405	0.0405 (0.0341, 0.0484)	0.0408 (0.0343, 0.0489)
μ^2/σ^2	595.8640	626.0555 (426.1530, 862.1156)	616.0582 (417.5656, 850.5031)

of medical laboratory technology. In the specialty of haematology and serology, two whole blood samples were sent to participants for measurement of Hb, RBC, MCV, Hct, WBC and Platelet in each survey. Fung and Tsang (1998) performed two parametric tests and the non-parametric test for equality of coefficients of variation with the measurement of Hb, RBC, MCV, Hct, WBC and Platelet in 1995 and 1996 survey. They showed that the coefficient of variation in 1995 is not significantly different from that of 1996 for the measurement of RBC, MCV and WBC in normal samples using two parametric tests and the non-parametric test. For this data sets, we only use the MCV data, and for MCV data with $n = 63$, the mean and standard deviation are 84.13 and 3.39, respectively.

We want to estimate $\theta_1 = \sigma/\mu$ and $\theta_1 = \mu^2/\sigma^2$. The results of the Bayes estimates and the 95% credible intervals are given in Table 4.5. For the computation of the Bayes estimates and the Bayesian credible intervals, we use 10 independent chains with a sample of size 20,000 discarding the first 10,000. Also the estimates by Withers and Nadarajah (2013) are given in Table 4.5.

For the case of $\theta_1 = \sigma/\mu$, the estimate by Withers and Nadarajah (W-N), the Bayes estimates and the credible intervals based on Jeffreys' prior and the matching prior have almost the same results. Since the estimates for the coefficient of variation are 0.0405 or 0.0408, we know that the variation of MCV data is very small. But for the case of $\theta_1 = \mu^2/\sigma^2$, the estimate by W-N and two Bayes estimates are a little bit different. The credible intervals based on Jeffreys' prior and the matching prior give slightly different results. For the lengths of credible intervals, Jeffreys' prior gives the small length and vice versa. However from results of our numerical simulation, we investigate that the matching prior gives the good coverage probabilities than Jeffreys' prior.

5. Concluding remarks

In normal distribution, a first order matching priors and Jeffreys' prior for the powers of mean and variance have been derived. We revealed that Jeffreys' prior and the first order matching prior have the different forms, that is, Jeffrey's prior is not the first order matching prior. Also we find the condition that the posterior distributions under the general priors are proper. From our numerical results, we show that the matching prior has a good performance rather than Jeffreys' prior in terms of the asymptotic frequentist coverage probability and the confidence interval.

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